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Notes on heavy cycles in weighted digraphs ☆

1. Introduction

We use [1] for terminology and notation not defined here, and consider digraphs containing no multiple arcs only.

Let D be a digraph. The numbers of vertices and loops of D are denoted by $n(D)$ and $r(D)$, respectively. We call D a *weighted digraph* if each arc a of D is assigned a nonnegative number $w_D(a)$, called the *weight* of a. For a subdigraph H of D, $V(H)$ and $A(H)$ are used to denote the set of vertices and arcs of H, respectively. The *weight* of H is defined by

$$w_D(H) = \sum_{a \in A(H)} w_D(a).$$

For a vertex $v \in V(D)$, $N_H^+(v)$ denotes the set, and $d_H^+(v)$ the number, of vertices in H to which there is an arc from v. We define the *weighted outdegree* of v in H by

$$d_H^{w+}(v) = \sum_{h \in N_H^+(v)} w_D(vh).$$

When no confusion occurs, we will denote $w_D(a)$, $w_D(H)$, $N_D^+(v)$, $d_D^+(v)$ and $d_D^{w+}(v)$ by $w(a)$, $w(H)$, $N^+(v)$, $d^+(v)$ and $d^{w+}(v)$, respectively.

An unweighted digraph D can be regarded as a weighted digraph in which each arc a is assigned weight $w(a) = 1$. Thus, in an unweighted digraph, $d^{w+}(v) = d^+(v)$ for every vertex v, and the weight of a subdigraph is simply the number of its arcs.

A loopless digraph is one containing no loops. Let D be a loopless digraph such that every vertex of D has outdegree at least d. It is easy to see that D contains a directed cycle with length at least $d + 1$. For weighted digraphs, Bondy [2] conjectured that if every vertex in a weighted loopless digraph has weighted outdegree at least 1, then the digraph contains a directed cycle of weight at least 1. This conjecture was disproved by Spencer of Nebraska (see [3]).

Bollobás and Scott [3] gave a lower bound on the weight of heaviest directed cycles in a weighted loopless digraph under the weighted outdegree condition.

Theorem 1 (*Bollobás and Scott [3]*). *Let D be a weighted loopless digraph with $n \geq 2$ vertices. If $d^{w+}(v) \geq 1$ for every vertex $v \in V(D)$, then D contains a directed cycle C such that $w(C) \geq (24n)^{-1/3}$.*

For an upper bound, Bollobás and Scott constructed a class of digraphs with minimum weighted outdegree at least 1 such that the maximum weight of cycles in these digraphs is at most $c \log_2 \log_2 n / \log_2 n$, where c is a constant and n is the order of the digraph.

As remarked in [3], it seems likely that $n^{-1/3}$ is much too small. Bollobás and Scott proposed the following conjecture.

Conjecture 1 (*Bollobás and Scott [3]*). *Let D be a weighted loopless digraph with $n \geq 2$ vertices. If $d^{w+}(v) \geq 1$ for every vertex $v \in V(D)$, then D contains a directed cycle C such that $w(C) \geq 2/\log_2 n$.*

In this paper, we prove the conjecture up to a constant factor.

Theorem 2. *Let D be a weighted loopless digraph with $n \geq 2$ vertices. If $d^{w+}(v) \geq 1$ for every vertex $v \in V(D)$, then D contains a directed cycle C such that $w(C) \geq 1/\log_2 n$.*

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In fact, we can prove the following stronger assertion.

Theorem 3. *Let D be a weighted digraph with $n \geq 1$ vertices and r loops. If $d^{w+}(v) \geq 1$ for every vertex $v \in V(D)$, then D contains a directed cycle C such that $w(C) \geq 1/\log_2(n+r)$.*

We postpone the proof of Theorem 3 to the next section.

2. Proof of Theorem 3

We use induction on n.

If D has only one vertex, denote it by v. By $d^{w+}(v) \geq 1$, we have $A(D) = \{vv\}$, $w(vv) \geq 1$ and $r = 1$. Thus $C = vv$ is a directed cycle with weight at least 1. The result is true. Now, we suppose that D has $n \geq 2$ vertices and r loops.

Case 1. D is not strongly connected.

Let D' be a strongly connected component of D such that there are no arcs from $V(D')$ to $V(D) \setminus V(D')$. It is easy to see that $d^{w+}_{D'}(v) = d^{w+}_{D}(v) \geq 1$ for all $v \in V(D')$. By the induction hypothesis, there exists a directed cycle C in D' (and thus, in D) such that $w(C) \geq 1/\log_2(n(D') + r(D'))$. Clearly $n(D') \leq n$ and $r(D') \leq r$. Thus, we have $w(C) \geq 1/\log_2(n+r)$, and complete the proof.

Case 2. D is strongly connected.

Case 2.1. There exists a vertex z such that $zz \notin A(D)$.

By D being strongly connected, there exists at least one arc with head z. Let y be a vertex such that $yz \in A(D)$ and $w(yz) = \max\{w(vz) : vz \in A(D)\}$. Consider the digraph D' such that $V(D') = V(D)\setminus\{y\}, A(D') = A(D-y) \cup \{vz : vy \in A(D)\}$, and

$$w_{D'}(uv) = \begin{cases} w_D(uy) + w_D(yz), & \text{if } uy \in A(D) \text{ and } v = z; \\ w_D(uv), & \text{otherwise.} \end{cases}$$

Note that if $zy \in A(D)$, then $zz \in A(D')$, and $w_{D'}(zz) = w_D(zy) + w_D(yz)$.

For every vertex $v \in V(D')$, its weighted outdegree is not less than that in D. Thus, we have $d^{w+}_{D'}(v) \geq 1$ for all $v \in V(D')$. By the induction hypothesis, there exists a directed cycle C' in D' such that $w_{D'}(C') \geq 1/\log_2(n(D') + r(D'))$. Since $n(D') = n - 1$, and D' contains at most one loop more than D, we have $r(D') \leq r + 1$. Thus $w_{D'}(C') \geq 1/\log_2(n+r)$.

If C' does not contain the vertex z, then it is also a directed cycle in D with the same weight. Otherwise, let xz be the arc in C' with head z. If $xy \notin A(D)$, then C' is also a directed cycle in D with the same weight. If $xy \in A(D)$, let C be the directed cycle obtained from C' by replacing the arc xz with the path xyz; then C is a directed cycle in D of weight $w_D(C) = w_{D'}(C') \geq 1/\log_2(n+r)$.

Case 2.2. For every $v \in V(D)$, $vv \in A(D)$.

In this case, D has $r = n$ loops. And we need only prove that there exists a directed cycle in D with weight at least $1/\log_2(n+n) = 1/(1+\log_2 n)$.

If there exists a loop with weight at least $1/(1+\log_2 n)$, then we complete the proof. So we assume that every loop of D has weight less than $1/(1+\log_2 n)$.

Let D' be the digraph obtained from D by deleting all the loops. Then D' has n vertices and no loops, and for each vertex v in $V(D')$, we have

$$d^{w+}_{D'}(v) \geq 1 - \frac{1}{1+\log_2 n} = \frac{\log_2 n}{1+\log_2 n}.$$

It is easy to see that D' is strongly connected. Note that for every vertex $v \in V(D')$, $vv \notin A(D')$. Using the conclusion of Case 2.1, we can obtain that there exists a directed cycle C in D' such that

$$w_{D'}(C) \geq \frac{1}{\log_2 n} \frac{\log_2 n}{1+\log_2 n} = \frac{1}{1+\log_2 n},$$

and C is also a directed cycle in D with the same weight.

The proof is complete. $\square$

References

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